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ELECTROMAGNETIC FORM FACTORS, TCHEBICHEF POLYNOMIALS AND GENERALIZED ROSENBLUTH FORMULA

Elastic scattering of the ultra relativistic polarized electrons on atomic nucleus with arbitrary spin is considered. The covariant parameterization of the electromagnetic current for a particle with arbitrary spin is used. This parameterization is based on the Bargman - Wigner formalism for the description of arbitrary spin particles (atomic nuclei) with using Tchebichef polynomials of a discrete variable for determination of the "physical" electromagnetic form factors that are electric and magnetic multipole momenta in the Breit reference frame. The generalized Rosenbluth formula is obtained for the cross section of the ultra relativistic electron scattering in a laboratory system in terms of the "physical" electromagnetic form factors as well as the initial and final polarization characteristics of the electron.

Keywords: elastic scattering, relativistic energy, polarized electrons, electromagnetic form factors, Bargman - Wigner formalism, Tchebichef polynomials, Rosenbluth formula.

Introduction

Some polynomials related to the algebra of angular momenta were studied by A. Meckler [1].

The pioneer works of R. Hofstadter [2] concerning electron scattering on atomic nuclei and structure of the nucleons have stimulated many theoretical investigations of nucleons and nuclear electromagnetic form factors. Hofstadter's high-energy electron-scattering measurements [3] have demonstrated clearly the existence of deviations from point-nucleon scattering laws. These measurements have demonstrated also that electron-scattering method appears to offer great promise in unraveling the problems of nuclear size and shape and the internal dynamics of nuclei. M. Gourdin was first who introduced the so-called "physical" electromagnetic form factors that are electric and magnetic multipole momenta of nuclei of arbitrary spin in a special reference frame (Breit system) [4, 5]. It was demonstrated later that Gourdin's "physical" form factors can be covariantly presenting by using Tchebichef polynomials of a discrete variable [6].

The main purpose of this paper is to use the covariant description [6] of the electromagnetic current for an arbitrary spin particle for calculations of cross sections of the ultra relativistic polarized electron scattering on the particle (nucleus) with arbitrary electric and magnetic multipole momenta.

Multipole momenta and Tchebichef polynomials of a discrete variable

Let us consider the nucleus with arbitrary spin S in the rest system of reference. In nonrelativistic quantum mechanics the operators of electrical and magnetic multipole momenta of the nucleus can be presented in terms of the completely symmetrical traceless "multipole" tensors $S_{i_1 i_2 \dots i_l}$ ($l \le 2S$) (in units of $\hbar = c = 1$):

$$Q_{i_1 i_2 \dots i_\ell} = Q_\ell \frac{(2\ell)!(2S-\ell)!}{(2S)!(\ell!)^2} S_{i_1 i_2 \dots i_\ell}, \tag{1}$$

$$M_{i_1 i_2 \dots i_\ell} = M_\ell \frac{(2\ell)!(2S-\ell)!}{(2S)!(\ell!)^2} S_{i_1 i_2 \dots i_\ell}, \tag{2}$$

where Q_ℓ and M_ℓ denote quantities $Q_{z\dots z}$ and $M_{z\dots z}$, averaged over states with maximum spin projections S of spin operator $\hat{S}_z$ on z axis. Tensors $S_{i_1 i_2 \dots i_\ell}$ can be uniquely expressed through spin operators $\hat{S}_i$ [1]

$$\hat{S}_i \hat{S}_k - \hat{S}_k \hat{S}_i = i\varepsilon_{ikl} \hat{S}_l, \quad \hat{S}_i \hat{S}_i = S(S+1)\hat{I} \tag{3}$$

up to a factor fixed by the following condition: the construction $S_{i_1 i_2 \dots i_\ell} q_{i_1} q_{i_2} \dots q_{i_\ell}$ represents the ℓ-power polynomial of the scalar product $(\hat{\boldsymbol{S}}\boldsymbol{q})$ with the unit coefficient by $(\hat{\boldsymbol{S}}\boldsymbol{q})^\ell$, $\boldsymbol{q}$ is arbitrary vector. The polynomials

$$\frac{(2\ell)!}{\ell!} S_{i_1 i_2 \dots i_\ell} a_{i_1} a_{i_2} \dots a_{i_\ell} = \varphi_\ell(\hat{\boldsymbol{S}}\boldsymbol{a}), \tag{4}$$

where $\boldsymbol{a}$ is unit vector, were introduced by P. L. Tchebichef in 1859 [7]. Acting by the polynomial $\varphi_\ell(\hat{\boldsymbol{S}}\boldsymbol{a})$ on the wave function of a particle with the spin S and its projection m on a direction $\boldsymbol{a}$, we obtain as the eigenvalue Tchebichef polynomial $\varphi_\ell(m)$ of discrete variable m:

$$\varphi_\ell(\hat{\boldsymbol{S}}\boldsymbol{a})\ \Psi_S(m) = \varphi_\ell(m)\Psi_S(m). \tag{5}$$

These polynomials also specified by the spin S, i.e. $\varphi_\ell(m)$ always means $\varphi_\ell(m,S)$.

Note, that Tchebichef polynomials $\varphi_\ell(m)$[1] are not so popular as famous Tchebichef polynomials $T_n(x)$ and $U_n(x)$ and therefore, one needs to specify them in the explicit form [7], for instance,

$$\varphi_0(m)=1,$$

$$\varphi_1(m)=2m,$$

$$\varphi_2(m)=12m^2-\left[(2S+1)^2-1\right],$$

$$\varphi_3(m)=120m^3-6\left[3(2S+1)^2-7\right]m,$$

$$\varphi_4(m)=1680m^4-120\left[3(2S+1)^2-13\right]m^2+$$

$$+9\left[(2S+1)^2-1\right]\left[(2S+1)^2-9\right],$$

$$\varphi_5(m)=30240m^5-8400\left[(2S+1)^2-7\right]m+$$

$$+30\left[15(2S+1)^4-230(2S+1)^2+407\right]m.^3 \quad (6)$$

Covariant parameterization of the electromagnetic current for a particle with arbitrary spin

Now we can write a covariant expression for matrix elements of the electromagnetic current in the case when a particle is described by Bargman - Wigner equations [9] (we use the metric and other notations given in [10]):

$$J_\mu=(2\pi)^3\sqrt{4E_1E_2}\left\langle p_2\left|j_\mu\right|p_1\right\rangle=\frac{1}{\left[-(p_2+p_1)\right]^S}\sum_{\ell=0}^{2S}(i)^\ell\frac{(2S-\ell)!}{(2S)!(2\ell)!\ell!}2^\ell q^\ell(\overline{u}_2(p_2)\Big\{Q_\ell(q^2)(p_2+p_1)_\mu\varphi_\ell\left(\frac{S_\rho q_\rho}{q}\right)\ \Big\}+$$

$$+\frac{i}{\ell}\frac{\sqrt{-(p_2+p_1)^2}}{2m_N}M_\ell(q^2)\varepsilon_{\mu\nu\rho\sigma}(p_2+p_1)_\nu q_\rho\frac{\partial}{\partial q_\sigma}\varphi_l\left(\frac{S_\rho q_\rho}{q}\right)\Big\}(u_1(p_1)), \quad (7)$$

where p_2 and p_1 are the 4-momenta of final and initial particles, $q=p_2-p_1$, $Q_l(q^2)$ and $M_l(q^2)$ are "physical" form factors that are electric and magnetic multipole momenta of the particle in the Breit system of reference [4], where $\boldsymbol{p}_2=-\boldsymbol{p}_1=\boldsymbol{q}/2$, $E_2=E_1$, m_N is the mass of nucleus. Here

$$S_\rho=\frac{1}{2i}\varepsilon_{\rho\mu\nu\sigma}S_{\mu\nu}\frac{(p_2+p_1)_\sigma}{\sqrt{-(p_2+p_1)^2}}, \quad (8)$$

$$S_{\mu\nu}=\sigma_{\mu\nu}^{(1)}+\sigma_{\mu\nu}^{(2)}+\ldots+\sigma_{\mu\nu}^{(2S)}, \quad (9)$$

$$\sigma_{\mu\nu}=\frac{\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu}{4i}, \quad (10)$$

γ_μ are the Dirac matrices, upper indexes in Eq. (9) show the index of spin-tensor $u_1(p_1)$ i.e. $u_{1\alpha_1\alpha_2\ldots\alpha_{2S}}(p_1)$ or conjugate spin-tensor $\overline{u}_2(p_2)$ i.e. $\overline{u}_2^{\alpha_1\alpha_2\ldots\alpha_{2S}}(p_2)$ on which the matrix $\sigma_{\mu\nu}$ acts. $u_1(p_1)$ and $\overline{u}_2(p_2)$ are normalized by condition:

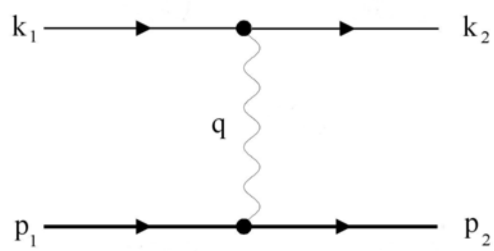
$$(\overline{u}\ (p)u(p))\equiv\overline{u}^{\alpha_1\alpha_2\ldots\alpha_{2S}}(p)u_{\alpha_1\alpha_2\ldots\alpha_{2S}}(p)=(2m_N)^{2S}. \quad (11)$$

The spin-tensor $u(p)$ satisfies Bargman - Wigner equations [9],

$$(ip_\mu\gamma_\mu+m_N)^{\alpha_i}u_{\alpha_1\alpha_2\ldots\alpha_i\ldots\alpha_{2S}}=0. \quad (12)$$

Cross section of elastic electron-nucleus scattering

Figure shows the Feynman diagram for an elastic electron-nucleus scattering.



One-photon exchange diagram for an elastic electron-nucleus scattering, 4-vectors k_1 and k_2 are 4-momenta of initial and final electrons, p_1 and p_2 are 4-momenta of initial and final nuclei.

The matrix element corresponding to this diagram is expressed through the leptonic $l_\mu=ie(\overline{u}_2(k_2)\gamma_\mu u_1(k_2))$ (electromagnetic current of

[1] A notation $\varphi_\ell(m)$ was introduced by Tchebichef himself [7], but in the modern mathematical literature [8] instead of them a little different polynomials are used, $t_\ell(S+m)=\frac{1}{\ell!}\varphi_\ell(m)$.)

electron) and the hadronic J_μ, see (7), (electromagnetic current of nucleus) currents [11]:

$$M = \frac{l_\mu J_\mu}{q^2}. \quad (13)$$

To calculate the cross section of the ultra relativistic polarized electron scattering on unpolarized nucleus it is necessary to calculate

$$\left\langle |M|^2 \right\rangle = \frac{l_\mu l_\nu^* \left\langle J_\mu J_\nu^* \right\rangle}{q^2}, \quad (14)$$

(in Eq. (14) l_ν^* and J_ν^* are complex conjugate l_ν and J_ν). The angle brackets in Eq. (14) denote the summation over polarizations of the final states of nucleus and the averaging over initial polarizations. The calculation of $l_\mu l_\nu^*$ gives (we neglect the electron mass m_e):

$$l_\mu l_\nu^* = (-\delta_{\mu\nu}(k_2k_1) + k_{2\mu}k_{1\nu} + k_{2\nu}k_{1\mu})(1+\sigma_2\sigma_1) +$$
$$+(\sigma_2+\sigma_1)\varepsilon_{\mu\nu\rho\sigma}k_{2\rho}k_{1\sigma} - -\delta_{\mu\nu}[(k_2k_1)(s_2s_1) -$$
$$-(k_2s_1)(k_1s_2)] + (k_2k_1)(s_{2\mu}s_{1\nu} + s_{2\nu}s_{1\mu}) +$$
$$+(s_2s_1)(k_{2\mu}k_{1\nu} + k_{2\nu}k_{1\mu}) - (k_1s_2)(k_{2\mu}s_{1\nu} + k_{2\nu}s_{1\mu}) -$$
$$-(k_2s_1)(k_{1\mu}s_{2\nu} + k_{1\nu}s_{2\mu}), \quad (15)$$

where σ_1 and σ_2 are longitudinal polarizations (helicities), s_1 and s_2 are transversal polarizations. We are considering the completely polarized electrons. We used the following expression for the density matrix of the electron [10, 12]:

$$u_\alpha \bar{u}^\beta = \left[\frac{(-ip_\mu\gamma_\mu)(1+i\gamma_5 s_\mu\gamma_\mu + \sigma\gamma_5)}{2}\right]_\alpha^\beta. \quad (16)$$

Note that frequently by calculating $l_\mu l_\nu^*$ one can neglect completely (and incorrectly) the transversal polarizations s_1 and s_2 of electrons [11]. The transversal polarizations of spin ½ massless particles are worked out in [10, 12]. Taking into account that the $\left\langle J_\mu J_\nu^* \right\rangle = \left\langle J_\nu J_\mu^* \right\rangle$ and $J_\mu q_\mu = 0$ one can choose (without the loss of generality) $(k_1s_2) = 0$ and $(k_2s_1) = 0$. Thus we obtain a new more simple expression for $l_\mu l_\nu^*$ equivalent to Eq. (15) ($k = (k_1 + k_2)/2$),

$$l_\mu l_\nu^* \equiv (-\delta_{\mu\nu}k^2 + k_\mu k_\nu)(1+\sigma_2\sigma_1 + s_2s_1) +$$
$$+k^2(s_{2\mu}s_{1\nu} + s_{2\nu}s_{1\mu}). \quad (17)$$

After calculations of $\left\langle J_\mu J_\nu^* \right\rangle$ as the final result we obtain the cross section of the ultra relativistic polarized electron scattering on nucleus of arbitrary spin in the laboratory system ($p = (p_1 + p_2)/2$),

$$d\sigma = \frac{1}{2}d\sigma_0 \left\{ [E(q^2) + M(q^2)][(1+\sigma_2\sigma_1 + s_2s_1 + \frac{(ps_2)(ps_1)}{m_N^2}\mathrm{tg}^2\frac{\theta}{2})] + 2M(q^2)(1+\sigma_2\sigma_1)(1+\frac{q^2}{4m_N^2})\mathrm{tg}^2\frac{\theta}{2} \right\}, \quad (18)$$

where θ is the scattering angle,

$$e^2E(q^2) = \sum_{\ell=0}^{2S} A_\ell q^{2\ell} Q_\ell^2(q^2),$$

$$e^2M(q^2) = \sum_{\ell=1}^{2S} \frac{\ell+1}{\ell} A_\ell q^{2\ell} M_\ell^2(q^2), \quad (19)$$

$$A_\ell = \frac{2^{2\ell}}{[(2S)!(2\ell)!]^2} \frac{(2S+\ell+1)!(2S-\ell)!}{(2S+1)(2\ell+1)} \quad (20)$$

and $d\sigma_0$ is the Mott cross section [13]

$$d\sigma_0 = e^4 \frac{\varepsilon_2}{\varepsilon_1} \frac{\cos^2\frac{\theta}{2}}{4\varepsilon_1^2 \sin^4\frac{\theta}{2}} d\Omega, \quad d\Omega = \sin\theta\, d\theta\, d\varphi, \quad (21)$$

ε_1 and ε_2 are the energy values of initial and final electrons.

Conclusions

Under the assumption of one photon exchange between the electron and the nucleus with arbitrary spin we have obtained the cross section of the ultra relativistic polarized electron scattering in the laboratory system in terms of the "physical" electromagnetic form factors as well as the initial and final helicities and transversal polarizations of electron. The role of Tchebichef polynomials of a discrete variable for calculations of the "physical" electromagnetic form factors was demonstrated. Our result in Eq. (18) is generalization of Rozenbluth formula [14] and its extention [4]. In [14] a formula is given for the cross section of elastic scattering of *unpolarized* electrons on unpolarized protons. In [4] a formula is given for the cross section of elastic scattering of *unpolarized* electrons on unpolarized nuclei of arbitrary spin. We have considered as distinct from [4] the scattering of *polarized* electrons on unpolarized nuclei of arbitrary spin.

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ЕЛЕКТРОМАГНІТНІ ФОРМФАКТОРИ, ПОЛІНОМИ ЧЕБИШЕВА ТА УЗАГАЛЬНЕНА ФОРМУЛА РОЗЕНБЛЮТА

Розглядається пружне розсіювання ультрарелятивістського поляризованого електрона на атомному ядрі з довільним спіном. При цьому використовується коваріантна параметризація електромагнітного струму частинки з довільним спіном, що ґрунтується на застосуванні формалізму Баргмана - Вігнера для опису частинок (атомних ядер) з довільним спіном та на використанні поліномів Чебишева дискретного змінного для визначення "фізичних" електромагнітних формфакторів, які являють собою електричні та магнітні мультипольні моменти у системі відліку Брейта. Одержано узагальнену формулу Розенблюта для поперечного перерізу розсіювання ультрарелятивістського електрона в лабораторній системі відліку. Ця формула виражена через "фізичні" електромагнітні формфактори ядра та через початкові й кінцеві поляризаційні характеристики електрона.

Ключові слова: пружне розсіяння, релятивістські енергії, поляризовані електрони, електромагнітні формфактори, формалізм Баргмана - Вігнера, поліном Чебишева, формула Розенблюта.

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ЭЛЕКТРОМАГНИТНЫЕ ФОРМФАКТОРЫ, ПОЛИНОМЫ ЧЕБЫШЕВА И ОБОБЩЕННАЯ ФОРМУЛА РОЗЕНБЛЮТА

Рассматривается упругое рассеяние ультрарелятивистского поляризованного электрона на атомном ядре с произвольным спином. При этом используется ковариантная параметризация электромагнитного тока частицы с произвольным спином, которая основывается на применении формализма Баргмана - Вигнера для описания частиц (атомных ядер) с произвольным спином и на использовании полиномов Чебышева дискретного переменного для определения "физических" электромагнитных формфакторов, которые являются электрическими и магнитными мультипольными моментами в системе отсчета Брейта. Получена обобщенная формула Розенблюта для поперечного сечения рассеяния ультрарелятивистского электрона в лабораторной системе отсчета. Эта формула выражена через "физические" электромагнитные формфакторы ядра и через начальные и конечные поляризационные характеристики электрона.

Ключевые слова: упругое рассеяние, релятивистские энергии, поляризованные электроны, электромагнитные формфакторы, формализм Баргмана - Вигнера, полином Чебышева, формула Розенблюта.

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